# Investigating the long-term evolution of galaxies: Noise, cuspy halos and bars

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**Abstract.** I review the arguments for the importance of halo structure in driving galaxy evolution and coupling a galaxy to its environment. We begin with a general discussion of the key dynamics and examples of structure dominated by modes. We find that simulations with large numbers of particles ($N \gtrsim 10^6$) are required to resolve the dynamics. Finally, I will describe some new results which demonstrates that a disk bar can produce cores in a cuspy CDM dark-matter profile within a gigayear. An inner Lindblad-like resonance couples the rotating bar to halo orbits at all radii through the cusp, rapidly flattening it. This resonance disappears for profiles with cores and is responsible for a qualitative difference in bar-driven halo evolution with and without a cusp. Although the bar gives up the angular momentum in its pattern to make the core, the formation epoch is rich in accretion events to recreate or trigger a classic stellar bar. The evolution of the cuspy inner halo by the first-generation bar paves the way for a long-lived subsequent bar with low torque and a stable pattern speed.

## 1. Introduction

Enormous effort in simulating the evolution of large scale structure over the last five years has led to a convincing scenario for galaxy formation and the early phases of evolution. Although these simulations are very far from a complete description of ISM and star formation, many of the basic features our in concert with observations. The final frontier is an understanding of evolution from formation to present. This is a very difficult problem for a number of reasons. First, the time scales are relatively long and span many characteristic orbital times. Second, there are multiple length scales involved: disk scale heights, scale lengths, dark-matter scale lengths. The evolution of galaxies almost certainly depends on the interaction between these scales.

My goal for this talk is to give you my views on the essential features that must be borne in mind when approaching this problem based on my own work and the work of others. I will emphasize stellar dynamics and implications for simulations. The talk will have two parts parts. I will begin with a review of the basic dynamics of galaxy structure (§2.). It is natural to think in terms of solving Newton's equations for $N$ particles. As long as this is done accurately, one only needs to interpret the simulation output. Of course, most of you are aware that this is naive and may easily lead to the wrong results. In the $N \to \infty$

limit, collective modes may dominate the structure of galaxies. This is obvious in disks where spirals and bars are apparent but true for halos as well, as I will describe. The majority of this talk will describe two scenarios: noise evolution (§§3.–4.) and bar-driven evolution[1] (§5.).

## 2.  Key features of stellar systems

Collisionless dynamics has a modal structure very different from everyday mechanical systems. A perfect elastic system (the textbook case of a drum-head or string) has infinite number of discrete modes. One may excite one mode at a time in principle. The situation is rather different for a stellar system. In general, stellar systems have an infinite number of continuous modes. Any real perturbation will excite infinitely many at a time. A wave-packet in a dispersive medium is a good analogy. The result is that nearly all perturbations to a galaxy damp. A practical example of these dynamics is *dynamical friction.* The familiar Chandrasekhar formula computes the change in momentum by accumulating the gravitational scattering between the perturbing mass and every other trajectory in the system. Alternatively, we may consider the response of the system to the moving body. The body passing through a stellar medium excites a wake. This wake exerts a gravitational tug on the body slowing it down. Relative to a particular center, the body's motion may be Fourier decomposed in to a broad continuous spectrum of frequencies, which couple to the system's modes at some multiple of these frequencies. Small multiples general lead to stronger excitation. The wake, then, consists of a superposition of continuous modes which quickly mixes away as the body continues on.

### 2.1.  Exceptions to damping

There are some notable exceptions to this strong damping. For example, one often sees spiral arms and bars in galactic disks. Why? Because the phase space is *squeezed* in its degrees of freedom. It is nearly a two-dimensional system with the vertical motion of acting independently of the radial and tangential ones. Similarly, the velocity dispersion is small compared to the mean tangential velocity. One can exploit the reduced dimensionality and disparate radial, tangential and vertical frequencies to find pattern speed which only couples weakly to the system's modes. In other words, the fewer the degrees of freedom, the fewer opportunities for coupling and damping. Bending modes are a good example. They are undamped for an infinitesimally thin disk but damped become damped as the thickness increases (Weinberg, 1991). The damping is caused by coupling between the longitudinal bending and the spectrum modes in with vertical motion. The larger the ratio of vertical to radial dispersion, the more disparate the frequencies and the weaker the damping.

Now what about the dark-matter halo? The halo is inhomogeneous in the radial direction: *squeezed* into its own gravitational bottle. This limits range of possible modal frequencies. Therefore just as the disk case, there are pattern speeds which weakly couples to the halo's continuum modes. Therefore, a halo
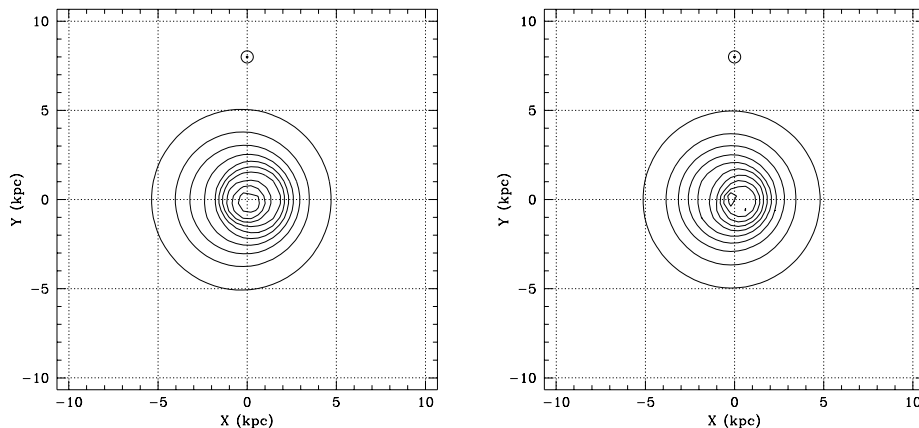
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[1]in collaboration with Neal Katz

Figure 1.    Prograde (left) and retrograde (right) modes for the a combined Toomre disk and $W_0 = 5$ King halo.

can exhibit a weakly damped response! Note that this situation is not true for an *infinite stellar medium.* The standard result that a *hot* stellar system will not exhibit patterns derives from the infinite stellar medium and is not correct for a real galaxy. More on this below.

## 2.2.   Weakly damped modes in a dark-matter halo

These weakly damped modes are a natural part of the dynamics for nearly all halos. I have found them in every model I have examined so far, both with and without cusps. For example, Figures 4 and 5 in Weinberg (1994) shows the mode shape and total density shape for an excited mode in a King $W_0 = 5$ model. This dipole mode shifts the center of the halo giving it a lopsided appearance. The pattern speed is non-zero but the damping time 1–2 orders of magnitude smaller than pattern speed!

It is natural to consider the disk separately in all of this. Of course, the disk and the halo are coupled gravitationally and together the two have a weakly damped modes. The rotation of the disk breaks the spherical symmetry and now there is a prograde and retrograde weakly damped mode. It won't surprise you to learn that the prograde mode damps more quickly than the retrograde mode but the shape of the modes are quite similar (see Fig. 1).

## 2.3.   Difficulty resolving modes in simulations

In order to see these modes in n-body simulations, the noise has to be small enough that the diffusion time for orbits is large compared to pass through the resonance (the libration time). This requires low-noise simulations. For this purpose, I use the expansion method (Clutton-Brock, 1972, 1973; Kalnajs, 1976; Fridman and Polyachenko, 1984; Hernquist and Ostriker, 1992). The use of orthogonal *wave functions* effectively filters out noise below the length scale of interest. These papers describe bases with density profiles that may not look like the underlying galaxy. A good fit with an arbitrary basis requires more

terms in the series. Each of these terms introduces another degree of freedom which adds to the total noise. Weinberg (1999) shows that one can choose a basis adaptively to match the equilibrium profile and the series then converges quickly and will result in low noise.

There are a number of advantages to this Poisson solver. It is *embarrassingly parallel* with $\mathcal{O}(N)$ scaling (see Hernquist et al., 1995). The coefficients may be computed on all processors simultaneously, for example, and therefore has none of the domain decomposition problems of a tree-code. Secondly, one can compare directly with perturbation theory which is one of my main attractions to this approach. For these reasons, it is very good choice for long-term evolution. The main disadvantage is that the basis can not adapt to arbitrary morphology so it is not good for mergers or following the non-linear development of instabilities where the end result is not a simple known geometry to start.

## 3.   Disk responding to noisy halo

Let's look at a simulation of an equilibrium King model halo with an exponential disk using this method. The halo concentration is chosen to match the characteristics of observed rotation curves. Scaled to the Milky Way, the halo core radius approximately the solar circle with a scale length of 3.5 kpc. The radial velocity dispersion is chosen large enough to stabilize the disk. The noise is dominated by an $m = 1$ disturbance and appears as a lopsidedness (as in Fig. 1). The orientation of the distortion varies with time but with the same shape. This is due to the excitation of the damped modes. The next order contribution to the noise, from $m = 2$, is smaller by an order of magnitude.

The expected noise amplitude due to excitation of modes can be computed analytically by perturbation theory and compared with n-body simulations. Figure 2 shows the power in each $l = 1$ basis function for a variety of simulations with different $N$ for the halo. The expected noise without self-gravity (Poisson noise, in other words) is shown for comparison (filled squares). For small $N$, the power in functions with small index $n$ is underestimated without self-gravity. The basis function index is equal to the number nodes; the scale structure is represented primarily by functions with small index $n$. At particle number $N \approx 10^6$, the simulations approach the theoretical expectation for small $n$. I conclude that one requires $\gtrsim 10^6$ particles to recover fine scale structure. I have used a low concentration model here; many more particles needed for higher concentration halos to cover the increased dynamic range. Also, the expansion method suppresses very small scales and reduce the total power in noise. Tree codes and other adaptive schemes (including direct summation) that can resolve a larger range of scales pay for this generality by needing a considerably larger $N$ to achieve the same results. There is a more general consequence of this restricted test: the dynamics of the noise-excited modes are fundamentally the same in any excitation and therefore large particle numbers will be required to correctly simulate any response (e.g. a similar value of $N$ required for convergence for simulations described in §5.). Conversely, if $N$ is an order of magnitude below the necessary number, some weakly damped modes may be completely missing from the response. Doubling $N$ may not change the the appearance of the simulation but this will not indicate convergence.
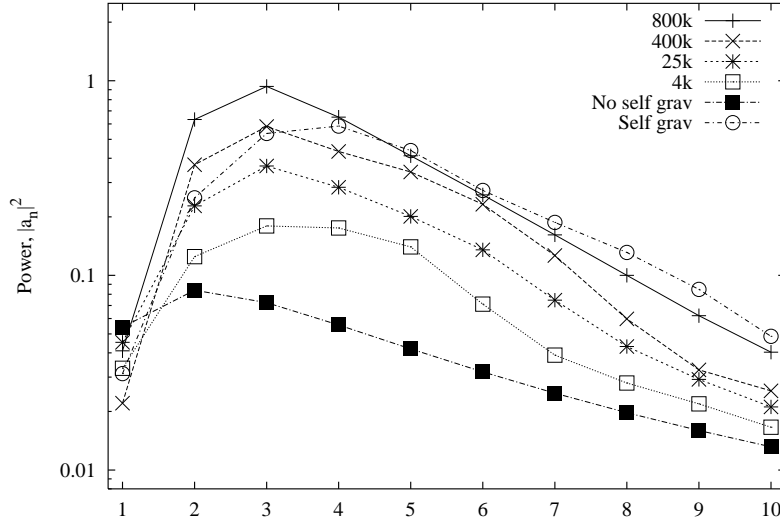
Figure 2.    Power from particle noise for simulations with varying numbers of particle $N$ (labeled). All are scaled to $N = 1$ and compared with the result from perturbation theory.

## 4.    Evolution of halo profiles by noise

The noise described in §3. can drive evolution in the halo. There are two limiting types of perturbations: point masses such as massive halo black holes and transient perturbers such as fly-by encounters, orbital decay, disrupting dwarfs. Point mass perturbers only drive halo structure at harmonic order $l > 2$ and self-gravity is less important in this high-$l$ regime[2]. The response to transient perturbations is dominated by $m = 1$ ($l = 1$) weakly damped modes. Because the weakly damped modes dominate the response, its appearance is only weakly dependent on details of perturbation. This can be demonstrated by direct example; the same profiles result from evolution by fly-bys and orbitally decaying point masses.

To compute the evolution, I use methods from stochastic differential equation. In short, assuming that individual perturbations are independent, the system reduces to an equation in Fokker-Planck form. One solves this Fokker-Planck equation for some time scale larger than the dynamical time but small compared to overall evolution time scale. Numerically, the solution procedure is analogous to that for Fokker-Planck evolution of globular clusters using operator splitting, etc. (Spitzer, 1987, for a review).

Let's look at some results of evolution due to transient perturbations. Figure 3 shows the halo profile after evolution due to decaying substructure for a variety of initial profiles: King models of varying concentration, the Plummer law and the cold dark matter profile suggested by Navarro et al. (1997, hereafter NFW).

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[2]More precisely, any quasi-periodic perturbation falls into this category. The only practical case that I can identify is orbital motion. However, some arbitrary applied periodic forcing could couple to low $l$.
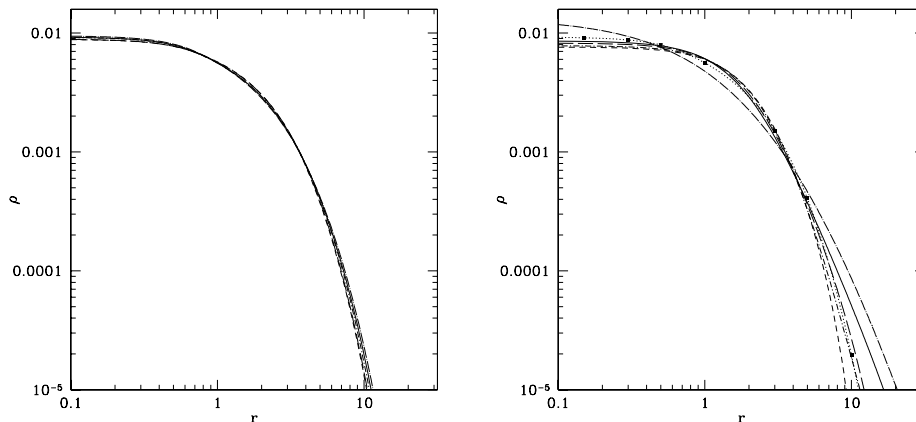
Figure 3. Evolved halo profiles for five different initial conditions: King models $W_0 = 3$ (dash), $W_0 = 5$ (long dash), $W_0 = 7$ (solid), Plummer (dash-dot) and double power law (long dash-dot) with $\gamma = 1$, $\beta = 4$ and $\epsilon = 0.1$. Left: the five profiles shown are compared after evolution. Right: the evolved profile are compared with the best fit to the initial profile. (These figures originally appeared in *Montly Notices of the Royal Astronomical Society.*)

As long the $m = 1$ modes are excited the noise looks nearly the same and the profile approaches the same form. Figure 3 compares both the evolved profiles and the fit of the original profiles to the evolved profiles. The inner part of the evolved profile can be acceptably fit by King models and Plummer models (which have cores) but not the NFW profile. At the same time, the all evolved profiles are similar, demonstrating the universality.

## 5.   Bar-driven evolution

Cold dark matter (CDM) structure formation simulations predict a universal cuspy halo profile (Navarro et al., 1997, NFW). This profile, $\rho \propto r^{-\gamma}(1+r/r_s)^{\gamma-3}$ or $\rho \propto r^{-\gamma}(1 + (r/r_s)^{3-\gamma})^{-1}$ with $\gamma = 1$, was first presented by NFW based on a suite of collisionless n-body simulations with different initial density fluctuation spectra and cosmological parameters[3]. Some observational evidence is consistent with such a cuspy dark matter profile (van den Bosch and Swaters, 2001) but not all (de Blok et al., 2001). Theoretically, disk bars and strong grand-design structure can strongly interact with the dark matter (Weinberg, 1985; Hernquist and Weinberg, 1992) by transferring angular momentum to the spheroid-halo component. This has been recently elaborated in a pair of contributions by Debattista and Sellwood (1998, 2000). They constructed a strong bar from an Q-unstable disk and following its evolution in a live halo. They found

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[3]Recent work debates the value of $\gamma$ (Jing and Suto, 2000; Moore et al., 1998, see) and but most estimates are in the range $1 < \gamma < 1.5$.

that the bars rapidly transferred angular momentum to massive (non-rotating) halos as predicted by Weinberg (1985). However, a few direct and wider variety of indirect observational inferences suggest rapidly rotating bars. Debattista & Sellwood conclude that dark-matter halos must have low density (or large cores) to be consistent with observations. This failure has potentially serious implications for the otherwise successful large-scale-structure paradigm.

After the first of these papers, Tremaine and Ostriker (1999) hypothesized that the interaction between tidal streamers and disk may flatten and spin up the inner halo-spheroid. The drag on bar reduced from flattened, rotating inner halo is then reduced. Another possibility is that there is something wrong with the dark matter hypothesis itself, perhaps requiring new physics (Spergel and Steinhardt, 2000).

Although this controversy has been illustrated by interesting and novel possibilities, dynamical evolution offers a natural solution in the context of CDM structure formation. In particular, simulations commonly produce bars in the gas-rich environment of a forming disk. The same torque which drives the slow down described by Weinberg (1985) and Debattista and Sellwood (1998) a provides sufficient torque to remove the inner cusp of an NFW profile soon after formation. At first look, this solution appears to run afoul with the observational evidence for rapid pattern speeds that began the controversy. However, this early gas rich bar is not the direct progenitor of the stellar bars seen in the present epoch. Moreover, the removal of the cusp paves the way for the later formation of a *classic* stellar bar with a stable or slowly decreasing pattern speed.

## 5.1. Physical description of the dynamics

It is easy to estimate that a strong bar can have important consequences for halo evolution. A toy model for a rotating gravitational quadrupole is two masses in orbit at the same distance from the center of a galaxy but at opposite position angles. We can use the dynamical friction formula to estimate the time scale for transferring all of the bar's angular momentum to the halo. The answer is: a few bar rotation periods (see Weinberg, 1985). The total angular momentum in the bar approaches that of the dark matter halo within the optical radius and therefore we expect that this torque can significantly change the inner halo profile.

This physical argument is gross simplification, however, and wrong in detail. The Chandrasekhar dynamical friction formula is derived by considering the momentum transfer in scattering of stars by a traveling body. However, a quasi-periodic halo orbit encounters the rotating perturbation many times. If the precession frequency leads or trails the pattern, the average net torque applied to this orbit will be zero. Kicks received at the exact commensurabilities between the orbital and pattern frequencies individual orbits break the symmetry and result in a density response that trails the bar. We illustrate this by an n-body simulation[4] of an NFW halo forced by a rotating bar (Fig. 4, left). This figure shows the halo density distortion in response to the bar. The bar's size and

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[4]The NFW halo is realized by $10^7$ equal mass particles and evolved using the expansion method described in Weinberg (1999). See §5.2. for additional detail.
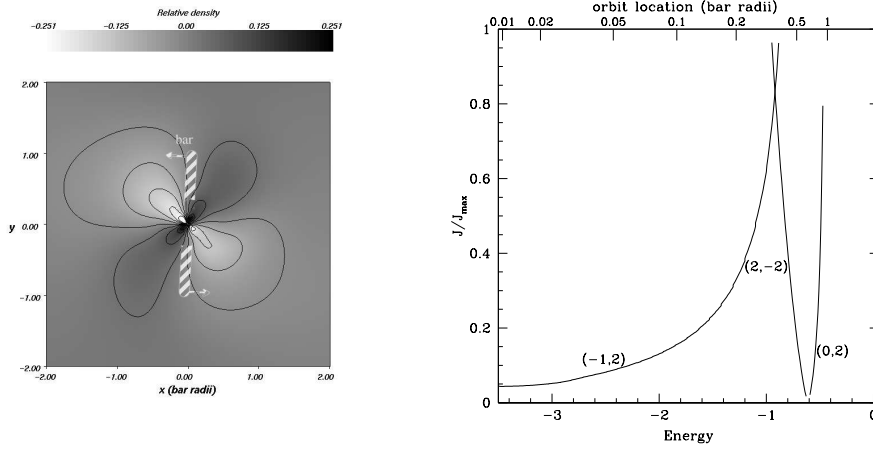
Figure 4. Left: the halo response and bar position during the evolution of an NFW-model halo. The mean density is subtracted and the amplitude of the resulting wake coded from underdensity (white) to overdensity (black). Right: location of low-order resonances in energy (lower axis) and characteristic radius (upper axis) for the bar in the NFW profile. The vertical axis describes the orbital angular momentum $J$ in units of the maximum angular momentum for a given energy, $J_{max}$. The inner-Lindblad-like resonance extends throughout the inner cusp. This resonance is absent from many dark-halo models with cores.

phase are shown schematically. The bar pattern leads the response pattern and therefore the bar torques the halo.

The location in the halo of the peak angular momentum transfer depends on halo profile itself in two ways. First, the torqued orbits will be near commensurabilities between the orbital frequencies and the pattern speed. These commensurabilities or *resonances* take form $l_r\Omega_r + l_\phi\Omega_\phi = m\Omega_{bar}$ where the three values $\Omega$ are the radial, azimuthal and pattern frequencies, respectively, $l_r$ and $l_\phi$ are integers and $m$ is the azimuthal multipole index. For the $l = m = 2$ multipole, we will denote a particular resonance by the tuple $(l_r, l_\phi)$. Low-order (high-order) resonances have small (large) values of $|l_r|$ or $|l_\phi|$. Second, a net torque requires a differential in phase-space density on either side of a particular resonance. If the bar is inside the halo core, the there will be very few nearby resonances and the torque is diminished. The dominant resonant orbits are at or beyond the core radius. It is the location of resonances that is key to a large bar torque rather than the dark matter density of the inner halo. Conversely, if the bar is in the dark-matter cusp, orbits near and inside the bar radius cover a large range frequencies with low-order resonances deep in the cusp. The torqued cusp orbits move to larger radii, decreasing the cusp density and decreasing the overall depth of the potential well. Both of effects cause the cusp to expand overall. Thus, the formation of a bar will naturally eliminate an inner cusp.

Because the bar–cusp couple depends on near-resonant dynamics, simulations with very high resolution will be required to resolve the dynamics, as previously described (§3.). The simulation shown in Figure 4 was the end-point of a

suite of simulations of increasing particle number from $10^4$ through $10^7$. Convergence using the expansion method was obtained for particle number $\gtrsim 4 \times 10^6$. Again, direct summation approaches (e.g. Kawai et al., 2000, GRAPE) and tree codes (Barnes and Hut, 1986) provide adaptive dynamic range but will most likely require higher particle number to obtain the same convergence. Note that even if the resonances are not resolved, these simulations will still exhibit significant torque. Our suite of simulations show that the overall torque *increases* as the particle number *decreases*! These same simulations give us good agreement with Chandrasekhar's formula. Such agreement is not a good indication of that one is observing the correct dynamics. Conversely, Chandrasekhar formula works well in simulations because resonances obliterated by artificial diffusion and therefore is well represented by scattering. In short, it is difficult to see resonant effects in n-body simulations because diffusion rate is high for moderate number of halo n-body particles.

## 5.2. Cusp removal in NFW profiles

*Methodology*   The evolution of the halo may be estimated analytically using a perturbation expansion of the collisionless Boltzmann equation and solution of the resulting initial value problem. The zeroth-order solution specifies the equilibrium galaxy and the first-order solution determines the forced response of the galaxy to some perturbation. The second-order solution determines the first irreversible change in the underlying distribution. For time scales much larger than an orbital time, the transients can be made arbitrarily small. Explicit comparisons with n-body simulations suggest that this approximation is acceptable even for a small number of orbital time scales. Mathematical details can be found in Weinberg (1985, 2001b).

The n-body simulations are performed using a parallel implementation of the algorithm described in Weinberg (1999) using the Message Passing Interface (MPI). This algorithm defines a set of orthogonal functions whose lowest-order member is the unperturbed profile itself. Each additional member in the series probes successive finer length scales. Because all scales of interest here can be represented with a small number of degrees of freedom, the particle noise is low.

The initial conditions are a Monte Carlo realization of the exact isotropic phase-space distribution function for the NFW profile, determined by Eddington inversion (see Binney and Tremaine, 1987, Chap. 4). The bar figure is represented by a homogeneous ellipsoid with axis ratios 1:0.5:0.05. We derive the bar force from the quadrupole part of the bar's gravitational potential. This rotating disturbance is turned on adiabatically over four bar rotation times to avoid sudden transients. The use of the quadrupole term only ensures that the dark-matter halo remains in approximate equilibrium as the bar perturbation is applied. We choose the corotation radius to be the NFW scale length and the bar radius is chosen to be 0.5 scale lengths. The results described below are only weakly sensitive to this choice.

*Results*   The linear theory demonstrates that the angular momentum transfer to the dark the matter takes place at commensurabilities between the bar pattern speed and halo orbital frequencies and is dominated by a few strong resonances. The torque pushes the inner halo orbits higher energy, removing gravitational
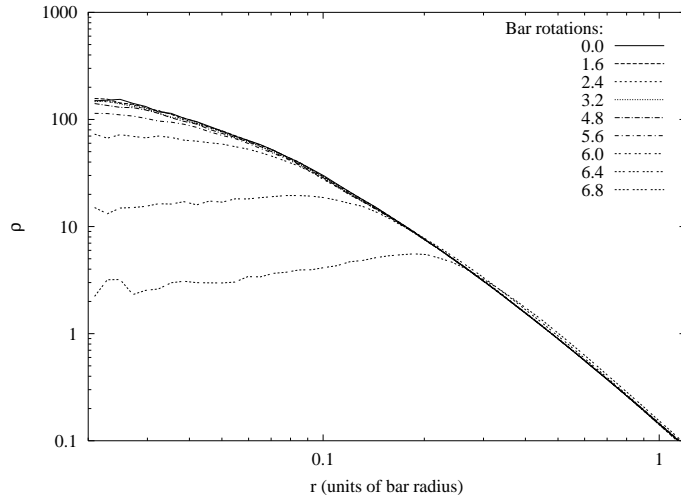
Figure 5.     Evolution of an NFW profile with an embedded disk bar whose length is $1/2$ of the scale parameter. Profiles are a time sequence labeled by number of bar rotation times.

support for the inner cusp. The halo expands and flattens the cusp. After a few bar rotation times (several hundred million years) the central cusp is clearly flattened. The n-body evolution, shown in Figure 5, is in agreement with the analytic predictions up to about five bar rotation times. At this point, approximately 30% of the available angular momentum in the body rotation has been transferred to the halo. Subsequent evolution is more rapid than predicted, presumably due to the feedback of the evolving profile on the near-resonant orbits although the details remain to be investigated; a similar super-linear increase in torque was reported by (Hernquist and Weinberg, 1992).

The linear theory allows us to explicitly identify the dominant resonances and the angular momentum transfer to halo orbits for models both with and without cores. The overall torque is dominated by one or two resonances in each case. For the NFW profile, the torque is dominated by the resonance $(-1, 2)$; this is inner Lindblad resonance analog. This resonance does not occur for a comparable King (1966) model. The first contributing radius for the King model is $(2, -2)$ near the location of the core radius. Higher-order resonances occur at larger radii are well-confined to a single characteristic radius (close to vertical in Fig. 4. However, the $(-1, 2)$ resonance is unique. It always exists in a cusp as $r \to 0$ because $\Omega_r \to 2\Omega_\phi$ as the orbital angular momentum $J$ approaches zero even though $\Omega_r$ $\Omega_\phi$ diverge for small $r$. Therefore there is always some value of $J$ for which $-\Omega_r + 2\Omega_\phi = 2\Omega_{bar}$. For this reason, the $(-1, 2)$ resonance can affect orbits deep within a cusp, dramatically changing the inner profile. For a model with a core, the core expands somewhat as angular momentum is transferred

toward the outer halo but otherwise remains qualitatively similar to its initial state.

## 5.3.  Discussion

Bar instability is ubiquitous in self-gravitating disks. Continued accretion and cooling is likely to precipitate a bar instability in the forming gas disk early on. Cosmological simulation suggest that these initial bars may be much larger than current-epoch bars. Once the rotating bar forms, its body angular momentum can be transferred to the halo as described above, flattening and removing the inner cusp. It is possible for this process to occur in stages with multiple gas bars; at each stage the inner core will grow. There will be a transition in the magnitude of the torque and halo evolution when the inner $(-1, 2)$ resonance is finally eliminated.

The underlying torque mechanism is not Chandrasekhar's dynamical friction but resonant angular momentum transfer. However, the noise on small scales will cause a high orbit diffusion rate with too few particles. The resulting scattering causes friction similar to that predicted by using the dynamical friction formula. Successful simulation of the resonant torque requires in excess of $10^6$ particles in the dark matter halo and perhaps considerably more depending on gravitational potential solver. My tests show that the torque is over-predicted in simulations with too few particles.

The presentation in §5.2. shows that the dominant resonance driving the cusp flattening is low order. Astronomical sources of noise such as orbiting substructure, decaying spiral waves, lopsidedness, etc. produce large-scale deviations from equilibrium that will not drive orbital diffusion in the inner halo within several bar rotation times to affect these predictions(see Weinberg, 2001b,a, for estimates of these time scales).

As the disk matures and becomes stellar- rather than gas-dominated, a normal stellar bar may form through secular growth or instability. The first-generation bar evolution will have diminished the inner-halo torque by flattening the cusp consistent with the Debattista and Sellwood (2000) arguments. We conclude that the observed lack of a central dark matter cusp in low surface brightness galaxies and dwarfs is a consequence of simple dynamical evolution and does not require a fundamental change to the nature of dark matter or galaxy formation. The difference in evolutionary end states may be the result of strong star-formation feedback evacuating the more weakly bound central potentials, lack of strong accretion events and mergers after the primordial bar has disappeared, or a combination of the two. Indeed, such stochasticity naturally predicts the inferred dispersion in present-day profiles.

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